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ABSTRACT

Research has demonstrated that people have difficulties recognizing ratio relationships involving multiplication. This paper is divided into four sections. The first section describes a study that was part of the three-year study of the mathematical and pedagogical development of prospective elementary teachers. The classroom teaching experiment involving 26 prospective elementary teachers employed constructivist teaching methodology to teach the concept of ratio-as-measure. Instruction was divided into two units. The first involved the development of understanding of the area of a rectangle as a multiplicative relationship between the sides. The second was a unit on the mathematical modeling of the concept of slope in order to learn to model situations with ratio. Researchers' analyses of videotapes of the classroom session led to the development of models of students' understanding. The second section presents and discusses three unanticipated difficulties encountered by the students in the exploration and discussion of slope: the notion of negative slopes, an inability to understand the relationship between the mathematical model and the situation that it models, and the inability to explore the validity of two methods presented to represent slope. The third section discusses the relationship between the students ability to understand the concept of ratio-as-measure and the larger concept of students' ability to understand mathematization, that is the process of mathematical modeling. In conclusion, mathematization of real world situations is a building block of the concept of ratio-as-measure. A suggestion is made for the development of mathematization throughout the mathematics curriculum. (18 references) (MDH)

Mathematization as a Component of the Concept of Ratio-as-Measure: A Study of Prospective Elementary Teachers

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Mathematization as a Component of the Concept of Ratio-as-Measure: A Study of Prospective Elementary Teachers

Background

The development of the concept of ratio and the development of proportional reasoning have intrigued researchers for decades (Inhelder & Piaget, 1958; Karplus, Karplus, & Wollman, 1974; Lesh, Post, & Behr, 1988; Noelting, 1980; Thompson, in press; Vergnaud, 1983). The concept of ratio is a key one in mathematics providing access to ideas as diverse as density, relative size, probability, slope of a line, similarity, and trigonometric ratios. Yet, ratio is poorly understood; difficulties with ratio concepts transcend national boundaries (Karplus, Karplus, Formisano, & Paulson, 1979).

The teaching of ratio often focuses on expressing relationships, such as 3 boys to 4 girls or 3 pounds per square inch. Instruction on proportion generally focuses on standard (algorithmic) strategies for finding a missing term. For example, for a problem such as "Bobby is making cookies which require 2 tablespoons of butter for 12 cookies. How much butter does he need for 18 cookies?", learners are taught a specific procedure for setting up and solving the proportion 2/12 = t/18.

Research has demonstrated difficulties that people have recognizing ratio (multiplicative) relationships (Hart, 1981; Inhelder & Piaget, 1958; Karplus, Karplus, Formisano, & Paulsen, 1979; and Karplus, Karplus, & Wollman, 1974). These studies demonstrate the regularity with which individuals select additive strategies when multiplicative strategies are appropriate. Our pretest results, based on open-ended written assessment and individual interviews, confirmed that such results were descriptive of our prospective elementary teachers as well.

Research, however, has not focused on the development of the ability to identify a ratio as the appropriate measure of a given attribute (hereafter referred to as "ratio-as-measure"). We emphasize two aspects of ratio-as-measure. First, is the need to distinguish between multiplicative and additive situations (as well as direct and indirect proportions[1]). The second emphasis, which seems to be absent from the research literature[2], is the need to recognize the ratio as the measure of the attribute in question. One of our pretest problems provides an example of a task that demands an understanding of ratio-as-measure. This problem involves recognizing that the "squareness" of a rectangle can be measured with the ratio width:length.

Pretest Problem: A new housing subdivision offers lots of three different sizes: 75 feet by 114 feet, 455 feet by 508 feet, and 185 feet by 245 feet. If you were to view these lots from above, which would appear most square? Which would be least square? Explain your answers.

This study focused on the development of ratio-as-measure.

Overview of the Project

The study was part of the Construction of Elementary Mathematics (CEM) Project[3], a three-year study of the mathematical and pedagogical development of prospective elementary teachers. The project studied these students[4] in the context of an experimental teacher preparation program designed to increase their mathematical knowledge and to foster their development of views of mathematics, learning, and teaching that were more consistent with the views espoused in recent reform documents (e.g., National Council of Teachers of Mathematics 1989; 1991). Data collection with 26 prospective elementary teachers proceeded throughout a mathematics course, a course on mathematics learning and teaching, a five-week pre-student-teaching practicum, and a fifteen-week student-teaching practicum.

The project was grounded in a social-constructivist view of learning (Cobb, Yackel, & Wood, in press; Ernest, 1991; Simon & Schifter, 1991). According to this view, mathematical understandings are individually and socially constructed. Individuals develop personally meaningful mathematical ideas as they participate in mathematical communities where a taken-as-shared mathematics is developed. The mathematical constructions of the individuals and of the group mutually contribute to and constrain each other.

The teaching approach of this project builds on the social-constructivist view of mathematics learning as well as on instructional emphases of current reforms (National Council of Teachers of Mathematics, 1991). Specifically the mathematics classroom is seen as a mathematical community in which mathematics is developed, examined, revised, and validated. The community is the judge of mathematical validity. Communication of mathematical ideas is critical to the operation of this community. Lessons are structured to challenge students to undertake problems which are related to big ideas in the domain of mathematics being studied. The exploration of the problems and the development of solutions take place, for the most part, in pairs or small groups. Ideas developed in the small groups are shared in the larger classroom community. Ideas are explained, justified, challenged, revised, and accepted. The

teacher's role is that of task developer, questioner, and facilitator of large group discussions. In addition, the teacher has a role in monitoring the understandings of the students involved and intervening appropriately. An assumption that we make is that the more completely the students take on the metacognitive load of a particular problem or task, the greater the probability that they will develop rich understandings, be able to interconnect their learnings, and apply them in novel situations.

The research on the mathematics course and the course on mathematics learning and teaching employed a constructivist teaching experiment methodology, as developed by Cobb and Steffe (1983) for research with individual subjects. We adapted that methodology to research on classroom mathematics (in the manner of Cobb, Yackel, and Wood, 1988). The first author taught all classes. Classes were videotaped and field notes were taken by project researchers. Following each class the first and second authors met to discuss what they inferred to be the conceptualizations of the students at that point and to plan for the next instructional intervention. These analyses and planning sessions led to the development of models of students' understanding.

The classroom teaching experiment in the mathematics course had as its specific focus the development of ratio-as-measure. It is uncommon for a classroom teaching experiment, which examines both the psychological and social aspects of the construction of mathematics, to be focused on the development of particular mathematical content. Pioneering work combining these two views of mathematics classrooms has been done by Cobb, Wood, and Yackel (in press). However, their focus has been on the nature of mathematical activity in the classroom and the development of an inquiry mathematics approach. Although they have described children's mathematical work, the development of particular mathematical content has not been the primary focus of their research.

The pretest problem presented above is an example of the type of problem which we wanted prospective elementary teachers to be able to solve by the end of the course. Consistent with prior research, the majority of our students (19 of 26) used an additive strategy (the difference between the two sides) rather than a ratio (width:length) to solve this problem at the outset of the course.

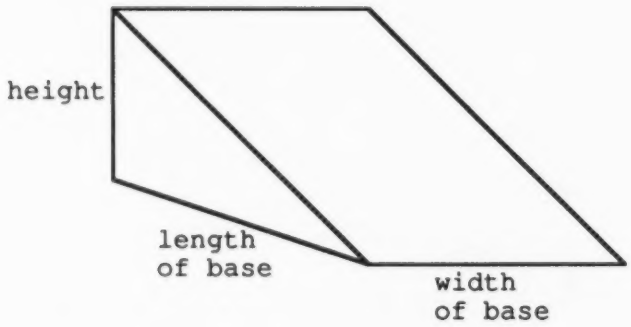
The Study

Our instruction was divided into two units. The first involved the development of understanding of the area of a rectangle as a multiplicative relationship between the sides (Simon and Blume, 1992). The second was a unit on mathematical modeling of

real world situations. The purpose of this second unit was for students to learn to model situations with ratio when appropriate.

Our goal was to contribute to a theory of how ratio-as-measure is learned. In lieu of such a theory, we proceeded with the strategy of engaging the students in problems in which ratios were appropriate measures and intermixed these problems with problems that involved other types of mathematical models (e.g., additive, inverse proportion). The following problem is an example of the former:

> In Kansas, there are no mountains for skiing. An enterprising group built a series of ski ramps and covered them with a plastic fiber which permitted downhill skiing. It is your job to rate them in terms of most steep to least steep. You have available to you the following measurements for each hill: the length and width of the base (measures along the ground) and the height. See the diagram. How would you determine the relative steepness using the information that you have?



Developing Mathematical Models

The ski slope problem led to ongoing discussions of representing steepness in two ways, by the ratio of the height to the length of the base or by the difference between the height and the length of the base. The class came to refer to these two proposed representations for steepness as the "ratio method" and the "difference method" respectively.

Karen: When we did the problem last time [in our small group] we talked about ratio as one way of looking at the numbers. But we also talked about if we subtracted those numbers instead of divided those numbers.

Candy: What are you doing when you're subtracting those numbers?

Karen: Just subtracting those. And do they help tell me steepness? ... Then we did the ratio [method] to see if it was true. ... We're just talking about subtraction right now --

using a subtraction method. ... So, the question is asking us to do, is explore what it means to have a steepness of two by the difference method. ... To have a steepness of two would mean that the difference between the two sides is two, right?

Bobbie: Yeah.

Karen: In the subtraction method.

Georgia: ... and with the rest of the slopes we would have found negative numbers if we would have used the difference method ...

Unanticipated Difficulties

As students engaged in exploration and discussion of these representations for steepness, they encountered difficulties which we had not anticipated. Three of these are discussed below.

What does a negative difference indicate?

One type of argument that was advanced by many of the students was that the difference method was not useful because it would give "negative slopes."

Georgia: If you had a negative number, then the hill would be in the ground ... and with the rest of the slopes we would have found negative numbers if we would have used the difference method, and negative numbers, I don't know, we just can't have negative slopes. I don't know, maybe in Kansas they have negative slopes, but, I mean, I don't know, it just doesn't make sense to me to have a negative slope.

Sara: All hills are going to have a slope, even if it's one or a hundred thousand or whatever, so no matter what, you're going to have a slope of at least one, and it can't be a negative number.

Instructor: Why?

Sara: Cause if it's negative, then there's not a slope.

Many of the students insisted that negative slopes either had no meaning, would indicate that the ski slope went into the ground, or would mean that you had to ski on the underside of the slope, grounds for dismissing the difference method. This line of thought proved to be very persistent.

This discussion of "negative slopes," seemed to suggest that the students did not have an appropriate basis for determining the meaning of the symbolic mathematical expression, h - l. Instead of interpreting a negative difference as an indication of the relative sizes of the numbers being compared (i.e., a negative difference indicates that

the minuend/height is larger than the subtrahend/length), the students used their past and present associations with the language "negative slope" to attribute meaning to the expression. For these students, the connection between a physical situation and the mathematical expression used to represent the quantitative relationships in that situation was not well-understood.

Which numbers should we compare?

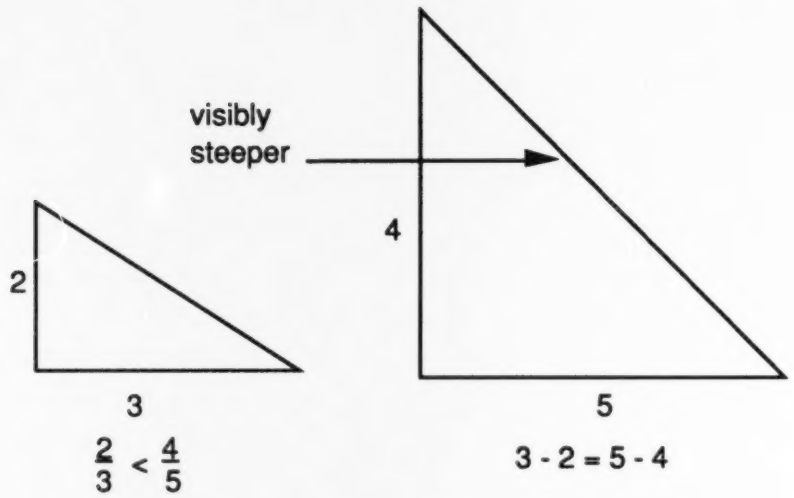
The students working on the ski slope problem seemed to lack heuristics for generating useful pairs of sides to compare. Conspicuously absent from the small and large group explorations of the problem were, "Let's work with two slopes that we know are the same steepness." or "Let's examine different slope dimensions that result in the same ratio (or the same difference)." Such strategies would suggest an understanding of the concept of invariance. Invariance refers to those aspects of a mathematical situation that do not change (remain invariant) while other aspects are varied. Harel, Behr, Post, and Lesh, (1991) identify an understanding of multiplicative invariance as an essential part of the understanding of ratio (i.e., that the ratio remains constant as the two quantities being compared vary). However, the idea of invariance extends well beyond multiplicative invariance; in discussing the ages of two siblings, the difference in their ages is invariant as their ages change.

Returning to our students' work with the ski slope problem, we observe that they do not seem to focus on invariance in the ski slopes or invariance in the mathematical formulations (i.e., ratio and difference). We see this as an indication that they do not understand the isomorphism between a mathematical model and the situation that it models. By "isomorphism," we wish to indicate that invariance in the model should correspond to invariance in the signified situation, and variation in the model should be consistent with variation in the signified situation in its direction and amount.

Does It Work?

Another difficulty encountered in the discussion of these two methods for expressing steepness was students' inability to explore the question, "Do these methods work?" We anticipated that a deductive justification of the ratio method would be difficult for these students. However, we thought that students might experiment with the two methods to collect some evidence as to their usefulness. For example, we thought that they might generate numbers for the quantities in question, draw triangles to scale in order to compare the steepness of the hypotenuses, and then compare the result of their visual inspection to the results of computing using the ratio and the difference methods respectively (see Figure 1.).

Figure 1



Instead students took a pair of side lengths (seemingly chosen for their ease in computing) and calculated both the ratio and the difference. Some realized that they had no criteria for choosing one method over the other, while some said that the difference method did not work because it sometimes gave a result that differed from that of the ratio method.

Instructor: Has anybody in the group given us any evidence that one of these [ratio or difference] is not going to be useful?

Deb: Yeah, when you had the 3 and the 3. If you had 3 and 3 for the one [e.g., ratio of 3 to 1 and 9 to 3], and then if you did it the other way it was different. If you used formula A [ratio] it wasn't the same as if you used formula B [difference]. So, obviously, they both can't be right.

Instructor: Do we have any evidence to which one is wrong? Have we seen any yet?

Jonnie: The only evidence so far that shows me that subtraction is wrong is that it is being inconsistent. Whereas division is always being consistent. And, you know, like we were saying, one way like the first way we used it [ratio], if the height gets higher the base gets shorter, it works that way. It's being consistent. But when you started to change it, it became inconsistent.

Instructor: What do you mean inconsistent?

Jonnie: Um, like up there, like, you know it said, um, the inconsistency to me showed up when I did the four over one, and then the six over two, and then I subtracted. It came out inconsistent with A [ratio method].

Instructor: That B [difference method] wasn't consistent with A?
Jonnie: Yeah.

When confronted with the fact that they had not determined that the ratio method was valid, they were at a loss as to how to progress. They seemed to lack a strategy such as scale drawings for determining which is actually steeper. We see this as an indication that they did not see the value of empirical work to address this problem.

Understanding Mathematization

What do these three examples have in common? They seem to suggest that these students have a weak understanding of the process of mathematical modeling or mathematization[5] (National Council of Teachers of Mathematics, 1989). They do not understand the relationship between the symbolic expression (the mathematical model) and the signified physical situation, and consequently they are unable to generate appropriate strategies for assessing the usefulness of proposed mathematical models.

The process of mathematization is a generic aspect of mathematics, and as such, is not unique to the study of ratios. Indeed understanding mathematizing can begin in the primary grades as students begin to see situations which can be represented by the sum or difference of two whole numbers. These students seemed to lack an adequate understanding of mathematization.

Implications for the Study of Ratio-as-measure

What is significant is not that we have uncovered weaknesses in prospective elementary teachers' understanding of mathematization, but that there is a critical relationship between understanding mathematization and understanding ratio-as-measure. Ratio-as-measure involves the expression of a quantitative relationship between two quantities as a ratio. To understand fully the decision to represent the relationship by a ratio, the learner must understand the process of mathematization as well as the multiplicative aspects of ratio. To respond to the question, "What does it mean to say that this ratio represents this relationship?", one must understand more generally what it means for a mathematical expression to represent a physical relationship. Therefore, we assert that generic knowledge of mathematization is a requisite component of an understanding of ratio-as-measure. It is not possible to make sense of the appropriateness of a ratio for measuring a particular attribute without understanding more generally the relationship between mathematical models and the situations that they model.

The Larger Context

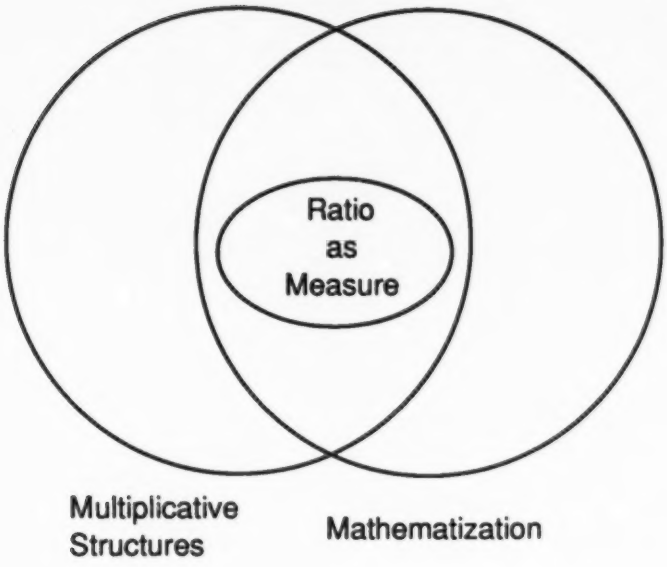
How does this description of the relationship between ratio-as-measure and mathematization differ from what is commonly discussed among adherents of the mathematics reform? Traditionally, the understanding of mathematical concepts, for example ratio, and the development of mathematical reasoning (including mathematization), are considered separately. A more progressive view is that mathematical processes run orthogonally to mathematics content; particular mathematics content provides opportunities for using and situating mathematical processes that need to be learned and these processes in turn are tools for exploring the content. This view still treats content and processes as separate categories, albeit categories of learning which may enhance each other. If one examines the *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics, 1989), the concept of ratio is discussed under "Standard 5: Number and Number Relationships," while mathematization is discussed under "Standard 1: Mathematics as Problem Solving." Although Standard 1 asserts "...problem solving...is a process by which the fabric of mathematics...is both constructed and reinforced," the discussion of ratio does not include any attention to ratio as a mathematical model.

Our data suggest that the concept of ratio-as-measure is not just enhanced by the use of mathematical processes but is dependent on an understanding of mathematization. Mathematization is a building block of the concept of ratio-as-measure much in the way that earlier understandings of multiplicative relationships are necessary to the concept.

The Overlap of the Two Domains

The previous discussion suggests the appropriateness of a view of understanding of ratio-as-measure as comprised of (but perhaps not limited to) an understanding of aspects of multiplicative structures and of mathematization. The data that follow seem to support this perspective. However, these data do not fall neatly into one of these two domains (i.e., difficulties cannot be attributed to either knowledge of multiplicative structures or of mathematization). Rather they seem to belong to their intersection (see Figure 2).

Figure 2



A question arose in the class during the exploration of the ski slope problem over whether the ratio height:length-of-base was useful in describing one ski slope or only useful in comparing two or more.

Tonie: I was just wondering, is the number we come up with when we use the ratio method relevant unless we're comparing it to other triangles?... Is that number that we get when we use the ratio method, is that even a relevant number unless we're comparing it with other slopes.

Instructor: In other words, if we said we had a slope of three, does that mean...

Tonie: We have that number of three, um okay that slope of three, right, does that mean anything unless we are comparing it to other slopes?

Molly: It's relevant in that you can find a triangle that will give you that from that number, like we got those numbers by comparing those two sides, okay? And if you have a three, you know that there's only, the way you got that three, if you use the ratio method was to divide the height by the length of the base. And you can come up with the triangle that will give you that value by having that three, so yeah, it does mean something, it means that's the relationship of the height to the base. The proportion.

Tonie: Right. I mean, I know it stands for something, but what is the relevance of it if it's not comparing it to other triangles?

Instructor: Let me dramatize this question. Okay? Here are my skis, I come

swaggering in, I just skied down a slope of 3. Pretty impressed, aren't you? Does it tell you anything?

Georgia: No...

Instructor: Tonie, was my dramatization a fair paraphrase of your question? Do you know how steep a slope I skied down?...How many say yes? [a few hands] How many say no? [more hands] Alright, we have enough difference of opinion to have an argument. [pause] Lilly?

Lilly: I think it doesn't matter if you know, because you have, like if you go back to the truck signs, something is um, or a ski slope, you know somebody has set up some kind of arbitrary rating so they can say, "okay that's got a 7% grade on it," that you know but that mountain range that was just rated is not being compared to another mountain range. You're going to go right down that mountain.

Instructor: So does a truck driver know what a 7% grade is?

Lilly: Well, somewhere, you know, somewhere there has been some arbitrary rating set up so that...

Lois: That's not true.

Lilly: Well, I mean somehow, if you're going down rapids and you go down class I rapids or you go down class 5 rapids, somebody has rated those so that if you're a boater you should know what those ratings are, if you're a truck driver, you should know what those ratings are, if you're a skier going down a slope, you'll know what those ratings are, somebody who has never done any of those things, those numbers might not mean anything to them.

Many of the students maintained for quite some time that the ratio was only useful in comparison to other ratios.. For these students it seems that the ratio is a "black box" that generates numbers which can be used to index a set of slopes. This reference to ratio as black box emphasizes that the ratio for these students did not represent a quantitative relationship between the two lengths. Rather the ratio discussed in this problem seemed to be as impenetrable as many of the formulae that these students had encountered in previous mathematics classes.

The conceptions involved in the students' black box view of the ratio seem to fall in the overlap between knowledge of multiplicative structures and knowledge of mathematization. With respect to knowledge of multiplicative structures, when confronted by the ratio 3:2, these students do not seem to be clear about the multiplicative relationship between the two quantities (i.e. that the height is 1.5 times the length of the base). They had not developed a concept of "rate," defined by Thompson (in press) as "a reflectively-abstracted constant ratio." With respect to knowledge of mathematization, there is a lack of expectation that the quantity that is generated (the

ratio) is meaningful, a reflection on the weak connection between the ratio and the situation that it represents.

In Conclusion

A number of questions can be raised as a result of this study and some implications can be drawn.

This research suggests that an understanding of mathematization of real world situations is a building block of the concept of ratio-as-measure. One might ask why previous studies have not revealed this type of connection between mathematical processes and mathematical content. We suggest two reasons why these connections are not usually observed. First, the mathematical content of instruction has not demanded these connections. When instruction on ratio and proportion focuses on writing ratios, as in "Write a ratio that expresses the number of boys to the number of girls in this classroom," or on finding the missing term in a proportion, a procedural emphasis, then mathematical modeling and validation of models are not essential or even relevant. In contrast, the ski slope problem was posed in such a way as to involve students in generating a mathematical model for the problem context. Second, if instruction is more traditional, predominantly lecture and demonstration, the mathematical processes of students are not challenged. An analogy can be drawn here: very different navigational skills are required of drivers finding their way in an unfamiliar city from those required of drivers who are equally unfamiliar with the city but who have a native in the passenger seat telling them where to turn. The learner in an inquiry classroom must have a fuller range of "navigational" skills. For most students in traditional classrooms, the mathematics that they need to solve the next set of "problems" is given to them. Students are generally not asked to develop mathematical formulations. As a result they are never faced with questions about the validity of the mathematics that they are dealing with. After all, the mathematics being used has been approved by "experts" (i.e., teachers, mathematicians, textbook authors).

The relationship we have postulated between mathematization and ratio leads to greater concern with the development of understanding of mathematization: How do students come to understand the process of mathematizing? How do students learn to make judgments about the validity of mathematical formulations? What are the mathematical abilities that allow students to explore mathematical relationships productively (i.e., generate useful examples, explore invariant relationships, etc.)? What other mathematical concepts are inaccessible due to lack of understanding of

mathematization?

An emphasis on ratio-as-measure changes some important aspects of pre-college mathematics. The results of this study suggest that ratio-as-measure is unlikely to be well understood if mathematization is not developed throughout the mathematics curriculum. If mathematical modeling is emphasized beginning in the primary grades, later explorations of ratio-as-measure are likely to be more productive. In addition we might wonder how students' mathematical understandings would be different if, instead of being handed prepackaged formulae and procedures, they explored the following questions:

1. By what process could we enlarge polygons while maintaining their shape?
2. How can we characterize the steepness of a line on a Cartesian graph?
3. What would be an appropriate way of representing the likelihood of an event, such as drawing a red cube out of a bag of cubes of different colors?

When the ratio is not given at the outset, as is typically done, but rather is developed by the classroom mathematics community as a useful mathematization of the situation, greater understanding of mathematization and of ratio are likely to result.

Finally, we wish to offer some implications for teacher education. Teachers must be provided with the opportunity to construct and develop thoroughly the mathematics which they will need to teach. (This is a minimum. Ideally, their mathematical knowledge should extend well beyond what they will teach.) This will not be accomplished by requiring that a higher level of mathematics be studied or that a greater number of traditional mathematics courses be taken. Many mathematics teacher educators (Committee on the Mathematical Education of Teachers, 1991; National Council of Teachers of Mathematics, 1991; Simon and Schifter, 1991) have advocated that teachers must be taught in ways that are consistent with how we would like them to teach. The reasons tend to be the potential for prospective teachers to develop greater conceptual understanding, to learn about learning and teaching as they learn mathematics, and to build useful conceptions of mathematics. These are strong reasons. However, this study suggests that the nature of the mathematical activity in the classroom (e.g. involvement in mathematical modeling) can also have a more direct effect on the learning of particular mathematical concepts (e.g. ratio-as-measure).

Footnotes

[1]We collected written and interview data on the following problem: "There is enough hay in the barn to feed 4 cows for 3 weeks. For how long could we feed 6 cows with the same amount of hay?" Prospective elementary teachers frequently chose to write the direct proportion 4 cows/3 weeks = 6 cows/x weeks.

[2]Readers are encouraged to call to our attention studies that deal with this issue.

[3]The Construction of Elementary Mathematics (CEM) Project is supported by the National Science Foundation under Grant No. TPE-9050032. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[4]In this paper, "students" refers to the prospective elementary teachers participating in the teaching experiment.

[5]"Mathematization" is the process of identifying and representing the quantitative relations (Thompson, in press) of a real-world situation. "Real-world" here denotes a context which includes more than quantitative information.

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